THE MIXED AREA OF A CONVEX BODY AND ITS POLAR RECIPROCAL*

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ABSTRACT

Half the vector sum of a convex body and its polar reciprocal with respect to a unit sphere E contains E. A consequence of this is: The mixed area of a plane convex body and its polar reciprocal with respect to E is minimized by circles concentric with E.

The arithmetic mean $(K + \hat{K})/2$ of a convex body K and its polar reciprocal $\hat{K}$, with respect to a unit sphere E centered at an interior point of K, contains E. From this we shall obtain the following result.

THEOREM. *The mixed area $A(K, \hat{K})$ of a plane convex body and its polar reciprocal satisfies $A(K, \hat{K}) \geqq \pi$, with equality if and only if K is a circle concentric with E.*

To prove that

$$(K + \hat{K})/2 \supseteq E \tag{1}$$

let Q be the center of E, x the boundary point of K in the direction v from Q. The polar plane of x has a normal distance from Q equal to $1/\| x \|$ where $\| x \|$ is the distance from Q to x. The normal distance to the support plane of K perpendicular to v is greater than or equal to $\| x \|$. Hence, if H and $\hat{H}$ are the support functions of K and $\hat{K}$ with respect to Q, we have, for the support function of $(K + \hat{K})/2$:

$$(H(v) + \hat{H}(v))/2 \geqq \| v \| \; (\| x \| + 1/\| x \|)/2 \geqq \| v \| \tag{2}$$

and the right hand side of (2) is the support function of E.

There is equality in (1) if and only if $\| x \| = 1$; therefore in the inclusion (1), with λK for K and $(\lambda K)^{\wedge} = \hat{K}/\lambda$ for $\hat{K}$ where $\lambda > 0$:

$$(\lambda K + \hat{K}/\lambda)/2 \supseteq E \tag{3}$$

there is equality if and only if λK is the unit sphere E.

The mixed volume $V(K_1, \ldots, K_n)$ is monotonic increasing in each convex

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body K_i, cf. [1]. We write $W_p(K)$ for the mixed volume with $K_1 = \ldots = K_p = E$ and the remaining K_i set equal to K. From (3) we have for $p < q$:

$$W_p([\lambda K + \hat{K}/\lambda]/2) \geqq W_q([\lambda K + \hat{K}/\lambda]/2),$$

with equality if and only if $\lambda K = E$, because in the case at hand the monotonicity is known to be strict, cf. [1], p. 43.

In the plane this yields

$$2A([\lambda K + \hat{K}/\lambda]/2) \geqq L([\lambda K + \hat{K}/\lambda]/2) \geqq 2\pi \tag{4}$$

since in this case

$$W_0(K) = A(K), \ W_1(K) = L(K)/2, \quad W_2(K) = \pi$$

where A and L are the area and perimeter.

From Steiner's formula we have

$$\begin{aligned} \min A([\lambda K + \hat{K}/\lambda]/2) &= \min[\lambda^2 A(K) + 2A(K, \hat{K}) + A(\hat{K})/\lambda^2]/4 \\ &\geqq (A(K, \hat{K}) + \sqrt{[A(K) A(\hat{K})]})/2, \end{aligned}$$

and

$$\min L([\lambda K + \hat{K}/\lambda]/2) = \min[\lambda L(K) + L(\hat{K})/\lambda]/2 \geqq \sqrt{[L(K) L(\hat{K})]},$$

the minima being taken over $\lambda > 0$. By Minkowski's inequality:

$$A(K, \hat{K}) \geqq \sqrt{[A(K) A(\hat{K})]}. \tag{5}$$

We replace the terms in (4) by these minima and use (5) to get

$$2A(K, \hat{K}) \geqq \sqrt{[L(K) L(\hat{K})]} \geqq 2\pi. \tag{6}$$

From the cases of equality in (3), we see that there is equality in (6) if and only if $K = rE$ for some $r > 0$.

In a similar fashion, in Euclidean 3-space we have, for the mixed surface area and total mean curvature

$$4\pi S(K, \hat{K}) \geqq \sqrt{[M(K) M(\hat{K})]} \geqq 16\pi^2,$$

with equality if and only if $K = rE$ for some $r > 0$.

Reference

1. Bonnesen, T. and Fenchel, W., 1934, *Theorie der konvexen Körper*, Berlin.

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